Quantum States via Random Tensor Method

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The random features method is a powerful technique for computing kernel matrices with reduced time and memory overhead. In these notes, we show that tensor contractions of one dimensional quantum states can be replaced by their averages, if matrices in the tensor decomposition are *incoherent.*

I. INTRODUCTION

Tensor networks are a standard method in representation of quantum states. In one dimension, tensor networks can be used to decompose eigenvectors of low entanglement as contractions of smaller tensors. Similar to how the Bethe-Anasatz provides a theoretical solution to a large class of 1d systems, tensor networks decomposition's have allowed for numerical solvablity of almost all one-dimensional problems of interest.

Despite work developing theories for two tensor methods for two-dimensional states, although (in the authors opinion) a fully satisfactory solution to the two-dimensional problem remains elusive, in this note, we suggest a few new tools that may help solve the 2d case. In this work, we show that replacing exact tensor contractions as contractions over random tensors can be used to reconstruct states with low-entanglement with high-fidelity. This is an interesting use case of concentration inequalities [1].

Our proposed random tensors approach is akin to the *quantum typicality* trick where the deterministic $\mathrm{Tr}[\mathcal{O}]$ operation can be well approximated by computing the inner product of $\mathcal{O}$ with a small number of randomly drawn states $|\Psi\rangle$. This phenomena is related to the concept of eigenvector de-localization. Eigenvector delocalization was observed in [2]. In someways, it is quite surprising that eigenvector delocalization has not been applied to tensor representations of quantum systems. The validity of the random tensor assumption hinges on property that tensors are incoherent in the standard Euclidean basis. For one dimensional systems, this can be proven using the canonical form of the PEPS.

We summarize our contributions as follows

- Akin to the quantum typicality trick, we show that exact contractions of Matrix Products States can be computed accurately as contractions of random tensors.
- We derive an explicit concentration bound on the error of this approximation.
- It has generally been understood that area low-entanglement of low-spectrum quantum states serves as. We suggest that similar representations can be found based on matrix *incoherence.*

A. Random Tensor Networks

Random tensor networks are of huge interest in quantum gravity [2]. Specifically, random tensor networks have been used as a tractable toy model of bulk-boundary correspondence [3]. By parameterizing the states as contractions of tensor, it is possible to develop field theories of low random states. It should be noted that in our work there is no randomness in the tensors $\Psi^{s_1 s_2 \dots s_k}$. We are replacing a totally deterministic tensor $\Psi^{s_1 s_2 \dots s_k}$ as an contraction of random tensors for computational purposes.

II. RANDOM TENSOR DECOMPOSITION

Let $\Psi^{s_1 s_2 \dots s_k}$ be a tensor carrying N-indices. The SVD can be made between the i-th and $(i+1)$-th index,

$$\Psi^{s_1 s_2 \dots s_k} = U_q^{s_1 s_2 \dots s_{i-1}} \Lambda_q \bar{V}_q^{s_1 s_2 \dots s_{i-1}}$$

where the matrix $\Lambda_q \geq 0$ has positive entries.

- Show a tensor diagram here

For notational simplicity, we will work with the exponential of the singular values

$$\Lambda_q = \exp(\lambda_q)$$

The random features method consists in replacing the *deterministic* tensor $\Psi^{s_1 s_2 \ldots s_k}$ with an expectation over *random* tensors

$$\Psi^{s_1 s_2 \ldots s_k} = \mathbb{E}_q[U_q^{s_1 s_2 \ldots s_{i-1}} \bar{V}_q^{s_{i+1} s_{i+2} \ldots s_k}]$$

where the random variable q is drawn from the distribution

$$\Pr[q] = \frac{\exp(\lambda_q)}{\sum_{q'} \exp(\lambda_{q'})}$$

Repeating this process,

$$\Psi^{s_1 s_2 \ldots s_k} = \mathbb{E}_{q_1 q_2 \ldots q_{k-1}}[U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} \ldots U_{q_{k-2} q_{k-1}}^{s_{k-1}} U_{q_{k-1}}^{s_k}]$$

where each q_i is drawn from the distribution

$$q_i \sim \frac{\exp(\lambda_q)}{\sum \exp(\lambda_q)}$$

An important quantity is the entropy of q_i random variable,

$$\mathcal{S}(q_i) = -\mathbb{E}_q[\log(q)]$$

we will see that this quantity, $\mathcal{S}(q_i)$, combined with the coherence $\mu(U)$ determines the validity of the random tensor approximation. The key observation is that each random variable q_i is independent. If each of the deterministic sub-tensors $U_{q_i q_{i+1}}^{s_i s_{i+1}}$ is 'spread out' in the Euclidean basis, then errors should self-average and the random variable $U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} \ldots U_{q_{k-2} q_{k-1}}^{s_{k-1}} U_{q_{k-1}}^{s_k}$ should be tightly concentrated around its mean. We can then evaluate terms of the form $\Psi^{s_1 s_2 \ldots s_k}$ by random sampling instead of exact contraction. Consider the cost of explicitly evaluating the tensor $\Psi_{s_1 s_2 \ldots s_k}$ on each index. The naive contraction method requires a summation over k B-dimensional bonds, and takes roughly $\mathcal{O}((d_1 + d_2 + \ldots + d_k)B)$ multiplications.

- There are 'smart' ways to contract matrices, how does this compare with existing method? Comment on complexity of exact computation vs random sampling

III. CONCENTRATION BOUNDS

The validity of the random tensor contraction approximation

$$\Psi^{s_1 s_2 \ldots s_k} = \mathbb{E}_{q_1 q_2 \ldots q_{k-1}}[U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} \ldots U_{q_{k-2} q_{k-1}}^{s_{k-1}} U_{q_{k-1}}^{s_k}]$$

depends on the concentration of the random variable $U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} \ldots U_{q_{k-2} q_{k-1}}^{s_{k-1}} U_{q_{k-1}}^{s_k}$ to its mean. Because each of the random variables q_i are independent random variables, we expect that errors self-average. Specifically, we aim to prove concentration bounds of the form

$$\Pr[\ |U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} \ldots U_{q_{k-2} q_{k-1}}^{s_{k-1}} U_{q_{k-1}}^{s_k} - \Psi^{s_1 s_2 s_3 \ldots s_k}]| \ \geq \epsilon\ ] \leq \exp(-\sigma^2 \epsilon^2)$$

for some $\sigma > 0$. If a bound of this form holds, then, the m-sample average

$$\Psi_m^{s_1 s_2 \ldots s_k} = \frac{1}{m} \sum_{j=1}^{m} U_{q_{kj} q_{1j}}^{s_1} U_{q_{1j} q_{2j}}^{s_2} \ldots U_{q_{(k-2)j} q_{(k-1)j}}^{s_{k-1}} U_{q_{(k-1)j} q_{kj}}^{s_k}$$

can be ϵ-close to the true average by taking $m \geq \frac{\sigma^2}{\epsilon^2}$ as

$$\Pr[\ |\Psi_m^{s_1 s_2 \ldots s_k} - \Psi^{s_1 s_2 s_3 \ldots s_k}]| \ \geq \epsilon\ ] \leq \epsilon)$$

holds for any choice of $m \geq \frac{\sigma^2}{\epsilon^2}$.

A. Single Random Index

Let us first consider the case of a single random index. Let

$$\Psi_{s_1 s_2} = \sum_{q=1}^{B} U_{s_1 q} \Lambda_q \bar{V}_{q s_2}$$

where B is the bond dimension of the tensor network. Let us define the random vectors $(U_q)_{s_1} = U_{s_1,q}$ and $(V_q)_{s_1} = V_{s_1,q}$. Using the random features method, we can write this as

$$\Psi_{s_1 s_2} = \mathbb{E}_q[U_{s_1 q} \bar{V}_{q s_2}] = \mathbb{E}_q[U_q V_q^\dagger]_{s_1 s_2}$$

where q is drawn from the exponential distribution of λ_q. This is an unbiased estimator as $\Psi_{s_1 s_2} = \mathbb{E}_q[U_{s_1 q} \bar{V}_{q s_2}]$

Let us assume that the matrices $U_{s_1 q}$ and $V_{s_1,q}$ are incoherent, so that

$$\mu(U) = ||U||_{2\to\infty} \leq \mu_U \sqrt{\frac{B}{d}} \quad \text{and} \quad \mu(V) = ||V||_{2\to\infty} \leq \mu_V \sqrt{\frac{B}{d}}$$

where d is the dimension of the Hilbert space and B is the bond dimension. Now, consider the generating function of the random variable $U_{s_1 q} \bar{V}_{q s_2}$,

$$\mathbb{E}_q[\exp(z U_{s_1 q} \bar{V}_{q s_2})] = \sum_{n=0}^{\infty} \frac{z^n}{n!} \mathbb{E}_q[(U_{s_1 q} \bar{V}_{q s_2})^n]$$

consider the term $\mathbb{E}_q[(U_{s_1 q} V_{q s_2})^n]$. We can bound each term in this expectation using the tensor product trick. We may write this as

$$(U_{s_1 q} V_{q s_2}^\dagger)^n = \mathrm{Tr}[(U_q V_q^\dagger \otimes U_q V_q^\dagger \otimes ... \otimes U_q V_q^\dagger)_{s_1 s_1 ... s_1, s_2 s_2 ... s_2}]$$

Thus, we have that

$$\mathbb{E}[(U_{s_1 q} V_{q s_2}^\dagger)^n] = \mathbb{E}[U_q V_q^\dagger \otimes U_q V_q^\dagger \otimes ... \otimes U_q V_q^\dagger]_{s_1 s_1 ... s_1, s_2 s_2 ... s_2}$$

By definition

$$\mathbb{E}[U_q V_q^\dagger \otimes U_q V_q^\dagger \otimes ... \otimes U_q V_q^\dagger] = \frac{1}{\sum_{q'=1}^{B} \exp(\lambda_{q'})} [\sum_{q=1}^{B} \exp(\lambda_q) U_q V_q^\dagger \otimes U_q V_q^\dagger \otimes ... \otimes U_q V_q^\dagger]$$

Thus, using the identity

$$||A \otimes B||_{2\to\infty} \leq ||A||_{2\to\infty} ||B||_{2\to\infty}$$

We have that,

$$\mathbb{E}[U_q V_q^\dagger \otimes U_q V_q^\dagger \otimes ... \otimes U_q V_q^\dagger]_{s_1 s_1 ... s_1, s_2 s_2 ... s_2} \leq \mu(U)^n \mu(V)^n$$

Thus, we have that

$$\mathbb{E}_q[\exp(z U_{s_1 q} \bar{V}_{q s_2})] \leq \exp(\mu(U)^2 \mu(V)^2 z^2)$$

Ergo, using basic properties of sub-Gaussian random variables,

$$\Pr[\ |\Psi_{s_1 s_2} - U_{s_1 q} V_{q s_2}^\dagger| \geq \epsilon\ ] \leq \exp(-\mu(U)^2 \mu(V)^2 \epsilon^2)$$

so that the random variable is sub-Gaussian with concentration parameter $\mu(U)\mu(V)$. Thus, if we assume that both U and V satisfy the bounded coherence condition

$$\mu(U) \leq \mu_U \sqrt{\frac{B}{d}} \quad \mu(V) \leq \mu_V \sqrt{\frac{B}{d}}$$

we have that

$$\Pr[\ |\Psi_{s_1 s_2} - U_{s_1 q} V_{q s_2}^\dagger| \geq \epsilon\ ] \leq \exp(-\frac{d^2}{\mu_U^2 \mu_V^2 B^2} \epsilon^2)$$

Note that the bond dimension is generically $B \sim d$. Thus, by taking $M \geq$ samples, we have that

$$\Pr[|\Psi_{s_1 s_2} - U_{s_1 q} V_{q s_2}^\dagger| \geq \epsilon] \leq \exp(\frac{-\epsilon^2}{\mu_U^2 \mu_V^2})$$

Ergo, the concentration properties are set by the coherence of the random matrices μ_U^2 and μ_V^2.

B. General Case

Now, that we have understood the single bond case, let us consider the general case. Suppose that the state Ψ can be decomposed as

$$\Psi^{s_1 s_2 \ldots s_k} = \sum_{q_1=1}^{B} \sum_{q_2=1}^{B} \ldots \sum_{q_{k-1}=1}^{B} U_{q_1}^{s_1} \Lambda_{q_1} U_{q_1 q_2}^{s_2} \Lambda_{q_1} U_{q_1 q_2}^{s_2} \ldots U_{q_1 q_2}^{s_2}$$

We can replace the deterministic tensor as a a contraction over random tensors. Via

$$\Psi^{s_1 s_2 \ldots s_k} = \mathbb{E}_{q_1 q_2 \ldots q_k}[U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} U_{q_3 q_4}^{s_3} \ldots U_{q_{k-1} q_{k-1}}^{s_{k-1}} U_{q_k}^{s_k}]$$

Now, consider the generating function of the random variable $U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} U_{q_3 q_4}^{s_3} \ldots U_{q_{k-1} q_{k-1}}^{s_{k-1}} U_{q_k}^{s_k}$. The n-th moment is given by

$$\mathbb{E}_q[(U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} U_{q_3 q_4}^{s_3} \ldots U_{q_{k-1} q_{k-1}}^{s_{k-1}} U_{q_k}^{s_k})^n]$$

Again using the tensor product trick, we may write

$$(U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} U_{q_3 q_4}^{s_3} \ldots U_{q_{k-1} q_{k-1}}^{s_{k-1}} U_{q_k}^{s_k})^n = \mathrm{Tr}[(U_{q_1} U_{q_1 q_2} U_{q_3 q_4} \ldots U_{q_{k-1} q_{k-1}} U_{q_k})^{\otimes n}]^{s_1 s_2 \ldots s_k}$$

Thus, we have that

$$\mathrm{Tr}[\mathbb{E}[(U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} U_{q_3 q_4}^{s_3} \ldots U_{q_{k-1} q_{k-1}}^{s_{k-1}} U_{q_k}^{s_k})^n]] \leq \prod_{i=1}^{k} \mu(U_k)^n$$

Ergo, we have that

$$\mathrm{Tr}[\mathbb{E}_q[\exp(z U_{q_1}^{s_1} U_{q_1 q_2}^{s_2} U_{q_3 q_4}^{s_3} \ldots U_{q_{k-1} q_{k-1}}^{s_{k-1}} U_{q_k}^{s_k})]] \leq \exp(\prod_{i=1}^{k} \mu(U_k) z^2)$$

so that the random tensor approximation is a sub-Gaussian random variable with concentration parameter less than the product of the coherence of $\mu(U_k)$. Thus, if each of the coherence's $\mu(U_k)$ are small, the random tensor approximation of Ψ will be tightly concentrated around the true value.

C. Incoherence of Generic Systems

Previous sections have shown that under the assumption of incoherence, tensor contractions can be replaced with expectations over random tensors. In this section we argue that generic states should be highly incoherent.

Consider a system of N sites, each of local dimension Hilbert space dimension d. Let $1 << i << N$ be a state near the middle of the chain. Consider the Schmidt decomposition of the state $|\Psi\rangle$

$$|\Psi\rangle = \sum_{i=1}^{\min(d^i, d^{N-i})} \Lambda_i |\Psi_L^i\rangle \otimes |\Psi_R^i\rangle$$

where $|\Psi_L^i\rangle$ are the left Schmidt eigenvectors and $|\Psi_R^i\rangle$ are the right Schmidt eigenvectors. Schmidt eigenvectors are orthonormal $\langle \Psi_i^R | \Psi_j^R \rangle = \delta_{ij}$. Consider the B-dimensional sub-space spanned by the right eigenvectors

$$\mathcal{P}_i^R = \sum_{i=1}^{B} |\Psi_R^i\rangle \langle \Psi_R^i| = PP^\dagger$$

We expect that this subspace is not localized in any of the local states $|s_{i+1} s_{i+2} \ldots s_N\rangle$. The states $|\Psi_i^R\rangle$ are of dimension d^i and so the projector $\mathcal{P}_i^R$ is to a subspace of dimension B. We conjecture that

$$\max_{x \in d^i} ||e_x^\dagger \mathcal{P}_i^L||_2^2 \sim C^R \sqrt{\frac{B}{d^i}}$$

where $C^R > 0$ is some constant. This statement is equivalent to saying that the eigenvectors are maximally 'spread out' among the basis states. Obviously, this (ref) does not hold for simple product states

$$|\Psi\rangle = \bigotimes_{i=1}^{N} |\Phi_i\rangle$$

but these product states have Schmidt values of one so that the random sampling method is actually *exact*.

- Specifically want to show that states with high-coherence are low-entangled states
- This suggest relation between entanglement and coherence?

Now, consider the canonical form of the matrix product states

$$|\Psi\rangle = \sum_{s_{i+1}s_{i+2}...s_k=1}^{d} U^{s_{i+1}}_{iq_{i+1}} \Lambda^{(i+1)}_{q_{i+1}} U^{s_{i+2}}_{q_{i+1}q_{i+2}} \Lambda^{(i+2)}_{q_2} ... U^{s_k}_{q_k} |s_1 s_2 ... s_k\rangle$$

Using this decomposition, the left-Schmidt eigenvectors are given by

$$|\Psi^R_i\rangle = \sum_{s_1 s_2 ... s_i=1}^{d} U^{s_{i+1}}_{iq_{i+1}} \Lambda^{(i+1)}_{q_{i+1}} U^{s_{i+2}}_{q_{i+1}q_{i+2}} \Lambda^{(i+2)}_{q_2} ... U^{s_k}_{q_k} |s_1 s_2 ... s_i\rangle$$

Using the randomization procedure, we may write

$$U^{s_{i+1}}_{iq_{i+1}} \Lambda^{(i+1)}_{q_{i+1}} U^{s_{i+2}}_{q_{i+1}q_{i+2}} \Lambda^{(i+2)}_{q_2} ... U^{s_k}_{q_k} = \mathbb{E}_q [U^{s_{i+1}}_{iq_{i+1}} U^{s_{i+1}}_{iq_{i+1}} ... U^{s_{i+1}}_{iq_{i+1}}]$$

Thus, the bounded coherence assumption (ref), is equivalent to the assumption that,

$$||U^{s_{i+1}}_{iq_{i+1}} \Lambda^{(i+1)}_{q_{i+1}} U^{s_{i+2}}_{q_{i+1}q_{i+2}} \Lambda^{(i+2)}_{q_2} ... U^{s_k}_{q_k}||_{2\to\infty} \leq C$$

Now, note that

$$U^{s_{i+2}}_{q_{i+1}q_{i+2}} \Lambda^{(i+2)}_{q_2} ... U^{s_k}_{q_k}$$

is a orthogonal matrix. Now, using the property $||AB||_{2\to\infty} \leq ||A||_{2\to\infty} ||B||_2$, we have that

$$||U^{s_{i+1}}_{iq_{i+1}} \Lambda^{(i+1)}_{q_{i+1}} U^{s_{i+2}}_{q_{i+1}q_{i+2}} \Lambda^{(i+2)}_{q_2} ... U^{s_k}_{q_k}||_{2\to\infty} = ||U^{s_{i+1}}_{iq_{i+1}} \Lambda^{(i+1)}_{q_{i+1}}||_{2\to\infty}$$

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V. COHERENCE

We review the concept of matrix coherence, first defined by [1]. Let $U \in \mathbb{C}^{n\times r}$ be a matrix with orthogonal columns, The coherence of U is defined as

$$\mu(U) = \frac{n}{r} \max_{i\in\{1,2,...,n\}} ||e_i^\dagger U||_2^2$$

Note that this the coherence defined vis-a-vis the Euclidean basis. If we view the columns of the matrix as a subspace of $\mathbb{R}^n$, then the projector $\Pi_U = UU^\dagger$ we have that

$$\mu(U) = \frac{n}{r} \max_{i\in\{1,2,...,n\}} ||\Pi_U e_i||_2^2$$

the coherence measures how 'spread out' the Euclidean basis vectors become when projected into U. Incoherence can also be defined with the $2 \to \infty$ norm,

$$\mu(U) = \sup_{||x||_2=1} ||Ax||_\infty$$

The bounded coherence property states that

$$||U||_{2\to\infty} \leq C\sqrt{\frac{r}{n}}$$

The coherence satisfies may nice properties. For example: Let $U \in \mathbb{C}^{n\times r}$ and $V \in \mathbb{C}^{n'\times r'}$ be two matrices with orthonormal columns. The tensor product $U \otimes V \in \mathbb{C}^{nn'\times rr'}$ then also has orthonormal columns. Consider the coherence of the tensor product of U and V,

$$\mu(U \otimes V) = \frac{nn'}{rr'} \max_{ij} ||(UU^\dagger \otimes VV^\dagger)(e_i \otimes e_j)||_2^2$$

Using the properties of the tensor product $||X \otimes Y||_2 = ||X||_2||Y||_2$, we have that

$$\mu(U \otimes V) = \frac{nn'}{rr'} \max_{ij} ||(UU^\dagger \otimes VV^\dagger)(e_i \otimes e_j)||_2^2 = [\frac{n}{r} \max_i ||UU^\dagger e_i||_2^2][\frac{n'}{r'} \max_j ||VV^\dagger e_j||_2^2] = \mu(U)\mu(V)$$

so that the coherence of the tensor product of two matrices is the product of each coherence.
Subjectivity The $2 \to \infty$ norm satisfies two important properties

$$||AB||_{2\to\infty} \leq ||A||_{2\to\infty}||B||_2, \quad ||CA||_{2\to\infty} \leq ||C||_\infty||A||_{2\to\infty}$$

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